

STABILITY OF THE SELF-SIMILAR DYNAMICS OF A VORTEX FILAMENT

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ABSTRACT. In this paper we continue our investigation about selfsimilar solutions of the vortex filament equation, also known as the binormal flow (BF) or the localized induction equation (LIE). Our main result is the stability of the selfsimilar dynamics of small perturbations of a given selfsimilar solution. The proof relies on finding precise asymptotics in space and time for the tangent and the normal vectors of the perturbations. A main ingredient in the proof is the control of the evolution of weighted norms for a cubic 1-D Schrödinger equation, connected to the binormal flow by Hasimoto's transform.

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1. INTRODUCTION

We consider the geometric PDE

$$(1) \quad \chi_t = \chi_x \wedge \chi_{xx}$$

that is usually known as the binormal flow (BF) or the localized induction equation (LIE). Above $\chi = \chi(t, x) \in \mathbb{R}^3$, x denotes the arclength parameter and t the time variable. Using the Frenet frame, the above equation can be written as

$$\chi_t = c b,$$

where c is the curvature of the curve and τ its torsion. This geometric flow was proposed by DaRios in 1906 [9] as an approximation of the evolution of a vortex filament in a 3-D incompressible inviscid fluid (see also [2]). We refer the reader to [1], [6], [29] and [24] for an analysis and discussion about the limitations of this model and to [28] for a survey about Da Rios' work. Local well-posedness results for the binormal flow were obtained when curvature and torsion are in high order Sobolev spaces, see [19, 23, 14]. A recent result of local well-posedness for the binormal flow, for less regular closed curves, was obtained by Jerrard and Smets [21, 22] by considering a weak version of the binormal flow.

The selfsimilar solutions with respect to scaling of (1) are easily found by first fixing the ansatz

$$(2) \quad \chi(t, x) = \sqrt{t} G\left(\frac{x}{\sqrt{t}}\right).$$

Plugging this ansatz in (1) and eliminating time one obtains the ODE

$$(3) \quad \frac{1}{2}G - \frac{s}{2}G' = G' \wedge G''.$$

After differentiation in s , calling $T(s) = G'(s)$, and using the system of Frenet equations we get

$$-\frac{s}{2}c n = -\frac{s}{2}T' = T \wedge T'' = c_s b - c\tau n,$$

where n denotes the normal vector. Hence we conclude that the selfsimilar solutions are characterized by the geometric conditions

$$c(s) = a, \quad \tau(s) = \frac{s}{2},$$

for a parameter $a \in \mathbb{R}$ (see [7]). The case $a = 0$ gives a straight line so that we can assume without loss of generality that $a > 0$. Given a , the corresponding solutions of (1) are unique modulo a translation and a rotation. Indeed, assume that the Frenet frame (T, n, b) at $s = 0$ is the identity matrix, so that from (3) we obtain $G(0) = 2a b(0) = (0, 0, 2a)$. Call G_a the corresponding curve and T_a its unit tangent. Hence we conclude that

$$\chi_a(t, x) = \sqrt{t} G_a\left(\frac{x}{\sqrt{t}}\right),$$

is a solution of (1) for $t > 0$ and that

$$T_a(t, x) = T_a\left(\frac{x}{\sqrt{t}}\right)$$

solves for $t > 0$

$$T_t = T \wedge T_{xx}, \quad |T| = 1,$$

usually known as the Schrödinger map onto the \mathbb{S}^2 sphere. We denote by

$$N_a(t, x) = (n_a + ib_a)(t, x) e^{i \frac{x^2}{4t}},$$

the “parallel” normal vector. The properties of this frame will be described in §2.

It was proved in [18] that there exist $A_a^\pm \in \mathbb{S}^2$ and $B_a^\pm \in \mathbb{C}^2$ such that for $x > 0$ (and similarly for $x < 0$),

$$(4) \quad \left| \chi_a(t, x) - A_a^+ \left(x + 2a \frac{t}{x} \right) - 4a \frac{t}{x^2} n_a(t, x) \right| \leq C \left(\frac{\sqrt{t}}{x} \right)^3,$$

$$(5) \quad |T_a(t, x) - A_a^+| \leq C \frac{\sqrt{t}}{x},$$

$$(6) \quad \left| N_a(t, x) - B_a^+ e^{ia^2 \log \frac{\sqrt{t}}{x}} \right| \leq C \frac{\sqrt{t}}{x}.$$

Moreover, $A_a^\pm \perp B_a^\pm$ and if we define θ as the angle between A_a^+ and $-A_a^-$

$$(7) \quad \sin \frac{\theta}{2} = e^{-\pi \frac{a^2}{2}}.$$

Also the coordinates of A_a^\pm and B_a^\pm are given explicitly in terms of Gamma functions involving the parameter a (see formula (55), (57), (47), (48), and (69) in [18]). In particular we can define at time zero for $x > 0$

$$(8) \quad \lim_{t \rightarrow 0} T_a(t, x) = A_a^+ \quad , \quad \lim_{t \rightarrow 0} N_a(t, x) e^{-ia^2 \log \frac{\sqrt{t}}{x}} = B_a^+,$$

and similarly when $x < 0$ using in that case (A_a^-, B_a^-) .

The reader can find in [18] some pictures of G_a and χ_a for different values of a . Also in [11] some numerical simulations are considered. In the figure 1.1 of that paper it is showed the remarkable similarity, at least at the qualitative level, of χ_a and the vortex filaments that appear in the flow of a fluid traversing a delta wind -see [26]. We also encourage the reader to look at the selfsimilar shape of the smoke rings in the picture 107 in [13]. It seems from these pictures and from the numerical simulations, that the selfsimilar dynamics of these vortex filaments are rather stable.

In our two previous papers [4] and [5] we obtain some results about the stability and the instability of the solutions χ_a . Our approach is based on the so-called Hasimoto transformation. In [19] the “filament function”

$$(9) \quad \psi(x, t) = c(x, t) \int_0^x \tau(s, t) ds,$$

is defined and it is proved that if c and τ are the curvature and the torsion respectively of a solution $\chi(x, t)$ of (1), then ψ solves the focusing cubic non-linear Schrödinger equation (NLS)

$$i\psi_t + \psi_{xx} + \frac{\psi}{2} (|\psi|^2 - A(t)) = 0$$

for some real function $A(t)$ that depends on $c(0, t)$ and $\tau(0, t)$. In the particular case of χ_a we have that for $t > 0$

$$(10) \quad \psi_a(x, t) = a \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}}$$

and $A(t) = \frac{|a|^2}{t}$.

Notice that

$$\int |\psi_a(x, t)|^2 dx = +\infty,$$

so that $L^2(\mathbb{R})$ is not the right functional setting to study ψ_a . It is natural to consider the so-called pseudoconformal transformation of ψ defining a new unknown v as

$$(11) \quad \psi(t, x) = \mathcal{T}v(t, x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} \bar{v}\left(\frac{1}{t}, \frac{x}{t}\right).$$

Then v solves

$$(12) \quad iv_t + v_{xx} + \frac{1}{2t}(|v|^2 - a^2)v = 0,$$

and $v_a = a$ is the particular solution that corresponds to ψ_a . A natural quantity associated to (12) is the normalized energy (see [3])

$$E(v)(t) = \frac{1}{2} \int |v_x(t)|^2 dx - \frac{1}{4t} \int (|v(t)|^2 - a^2)^2 dx.$$

An immediate calculation gives that

$$\partial_t E(v)(t) - \frac{1}{4t^2} \int (|v|^2 - a^2)^2 dx = 0,$$

and in particular $E(v_a) = 0$.

The binormal flow (1) is an equation that it is reversible in time. If we want to study perturbations of χ_a one possibility is to go forward in time starting at time $t = 0$ with a datum close to

$$(13) \quad \chi_a(0, x) = \begin{cases} A_a^+ x & x \geq 0 \\ A_a^- x & x \leq 0, \end{cases}$$

and to construct a solution up to say time $t = 1$. Another possibility is to give a datum at time $t = 1$ close to G_a and go backwards in time up to time $t = 0$.

In terms of v these two possibilities are rephrased as follows. First we write

$$v = a + u,$$

so that u has to be a solution of

$$(14) \quad iu_t + u_{xx} + \frac{a+u}{2t}(|a+u|^2 - a^2) = 0.$$

In particular

$$u(1, x) = e^{i\frac{x^2}{4}} \overline{\psi}(1, x) - a.$$

Then notice that the pseudoconformal transformation sends the interval of time $[0, 1]$ into the interval $[1, \infty)$. So that the first possibility, that is to say to go forward in time in (1), amounts to give a small asymptotic state at time infinity and construct a solution for $t \geq 1$ of (14) that remains close to it in an appropriate sense. The second possibility is to solve the initial value problem of (14) with some small datum at $t = 1$ and to prove the existence of a scattering state at infinity with a size controlled by that one of the initial datum. In [4] and [5] we study the two problems. Finally let us notice that long time asymptotics were studied for equations with a common point with (12) in terms of the nonlinearity, like the linear Schrodinger equation with a time depending potential (see ch.4 of [12]), the 1-D cubic NLS ([27],[8],[20]), the 2-D Gross-Pitaevskii equation ([17]), and the 2-D quadratic NLS ([25],[30],[15]). However the framework, approach and results for (12) are quite different.

More concretely in [5] we consider small initial data at time $t = 1$, $u_1(x) = u(1, x) \in X_1^\gamma$, $0 < \gamma < \frac{1}{4}$, where

$$(15) \quad \|f\|_{X_{t_0}^\gamma} = \frac{1}{t_0^{\frac{1}{4}}} \|f\|_{L^2} + \frac{t_0^\gamma}{\sqrt{t_0}} \| |\xi|^{2\gamma} \hat{f}(\xi) \|_{L^\infty(\xi^2 \leq 1)}.$$

In the Theorem 1.1 of that paper -see also section 3 below, we prove that there exists $f_+ \in L^2$ for which

$$\left\| u(t) - e^{i\frac{a^2}{2} \log t} e^{i(t-1)\partial_x^2} f_+ \right\|_{L^2} \leq \frac{C(a, u_1)}{t^{\frac{1}{4} - (\gamma + \delta)}} \|u_1\|_{X_1^\gamma} \xrightarrow[t \rightarrow \infty]{} 0,$$

for any $0 < \delta < 1/4 - \gamma$. Finally, the asymptotic state f_+ satisfies for all $\xi^2 \leq 1$ the estimate

$$|\xi|^{2(\gamma + \delta)} |\hat{f}_+(\xi)| \leq C(a, \delta) \|u_1\|_{X_1^\gamma}.$$

The main purpose of this paper is to prove that most of the properties (4)-(8) that describe the dynamics of the selfsimilar solution χ_a still hold under some extra conditions in u_1 . As a consequence we prove that there is a natural function space such that the selfsimilar dynamics going backwards in time remain stable under small perturbations of $G_a(x) = \chi_a(1, x)$.

Our main result is the following one.

Theorem 1.1. *Let $a > 0$ and let u_1 be a function small with respect to a in $X_1^\gamma \cap H^4$, with $0 \leq \gamma \leq \frac{1}{4}$, such that $xu_1, x\partial_x u_1 \in L^2$. Given $\chi_1(0) \in \mathbb{R}^3$ and $\partial_s \chi_1(0) \in \mathbb{S}^2$, let $\chi_1(x)$ be the corresponding curve with filament function $ae^{i\frac{x^2}{4}} + u_1(x)e^{i\frac{x^2}{4}}$ ¹. Then the unique Lipschitz solution $\chi(t, x)$ of the binormal flow for $0 \leq t \leq 1$ with $\chi(1, x) = \chi_1(x)$ constructed in [5] enjoys the following properties.*

¹this implies $u_1(0) \in \mathbb{R}$.

(i) **Asymptotics in space for the tangent vector and the normal vectors at fixed time:** There exist $T^{\pm\infty} \in \mathbb{S}^2$ and $N^{\pm\infty} \in \mathbb{C}^3$ such that for all $0 < t \leq 1$, and $x \neq 0^2$,

$$|T(t, x) - T^{\pm\infty}| \leq C(u_1) \left(\frac{1}{\sqrt{|x|}} + \frac{\sqrt{t}}{|x|} \right),$$

$$\left| N(t, x) - N^{\pm\infty} e^{ia^2 \log \frac{\sqrt{t}}{|x|}} \right| \leq C(u_1) \left(\frac{1}{\sqrt{|x|}} + \frac{\sqrt{t}}{|x|} + \frac{t}{x^2} \right).$$

(ii) **Further informations on the tangent vector:** For all $x \neq 0$ and all $t > 0$ small enough with respect to u_1 and $|x|$,

$$|T(t, x) - T^{\pm\infty}| \leq C(u_1),$$

and

$$\sin \frac{(T^\infty, -T^{-\infty})}{2} = e^{-\frac{a^2}{2}} (1 + C(u_1)).$$

(iii) **Formation of a corner at time 0:** For all $x \neq 0$

$$|\chi(0, x) - \chi(0, 0) - T^{\pm\infty} x| \leq C(u_1) |x|.$$

(iv) **Existence of a limit for the tangent at time 0:** For all $x \neq 0$ there is a limit for $T(t, x)$ as t goes to zero and

$$|T(t, x) - T(0, x)| \leq C(u_1) \left(\frac{\sqrt{t}}{|x|} + \frac{t}{x^2} + t^{\frac{1}{4}-} \right).$$

Moreover

$$T_x(0) \in L^1 \cap L^2.$$

(v) **The exact value of the angle of the corner:** The angle of the self-similar solutions is recovered at time 0,

$$\sin \frac{(T(0, 0^+), -T(0, 0^-))}{2} = e^{-\pi \frac{a^2}{2}}.$$

More precisely, modulo a rotation, we recover at the singularity point the self-similar structure

$$\lim_{x \rightarrow 0^\pm} \lim_{t \rightarrow 0} T(t, x) = A_a^\pm, \quad \lim_{x \rightarrow 0^\pm} \lim_{t \rightarrow 0} N(t, x) e^{-ia^2 \log \frac{\sqrt{t}}{x}} = B_a^\pm.$$

Remark 1.2. The above theorem gives a precise result about the dynamics of the perturbed filament in the selfsimilar region $|x| > \sqrt{t}$ for $1 \geq t \geq 0$. In particular it proves the existence of a natural binormal frame associated to the curve $\chi(0, x)$ even though it has a corner at $x = 0$. For doing this it is crucial to be able to use that $u(t)$ belongs to weighted L^2 spaces. All the analysis follows from the property that the tangent vectors of the perturbed filament are fixed for $x = \pm\infty$ and $1 \geq t > 0$. Once this is proved we integrate the Frenet frame, in fact we use the so-called parallel frame that turns out to be much more convenient, from $\pm\infty$ to $|x| > \sqrt{t}$. This is enough for our purposes.

²In the following the relations for $x > 0$ will involve $T^{+\infty}$ and $N^{+\infty}$ and the ones for $x < 0$ will involve $T^{-\infty}$ and $N^{-\infty}$.

Remark 1.3. *We do not obtain anything new in the interior region $|x| < \sqrt{t}$. At this respect we recall Theorem 1.4 of [4]. In that theorem it is proved that if the zero Fourier mode of the asymptotic state that determines $u(t, x)$ vanishes in an appropriate sense, then $\chi(t, x)$ remains close to $\chi_a(x, t)$ together with their respective Frenet frames also in the region $|x| \leq \sqrt{t}$. In particular the trajectory $\chi(t, 0)$ and the one of the frame $(T, n, b)(t, 0)$ remain close to $\chi_a(t, 0)$ and to the identity matrix. As a consequence, a very natural question is to characterize the asymptotic states of solutions $u(t)$ that belong to weighted L^2 spaces. It turns out that the answer is more delicate than what one could expect so that we will study it in a forthcoming paper. Finally, recall that in the appendix B2 of [5] it is proved that the zero Fourier modes of solutions $u(t)$ that are in weighted L^2 spaces typically grow logarithmically in time.*

The paper is organized as follows. In section 2 we introduce the parallel frame and its connection with the Frenet frame. The proof of our theorem is given in sections 3-5. In the appendix we show some estimates about the evolution in time of the norms of weighted L^2 spaces for the solutions $u(t)$ of (14).

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2. THE PARALLEL FRAME

In the original work of Hasimoto [19], for performing the transformation (9) a non-vanishing condition on the curvature was imposed. This condition has been removed by Koiso [23] who worked with another frame than the Frenet one. Although in our case the curvature does not vanish for small perturbations of the selfsimilar solutions, we shall take advantage of this Hasimoto-type link built between the cubic 1-D NLS and the binormal flow (1). We shall detail it below. The use of this frame makes the calculations of the next sections much shorter.

Given $a > 0$ we start with a solution of

$$(16) \quad i\psi_t + \psi_{xx} + \frac{\psi}{2} \left(|\psi|^2 - \frac{a^2}{t} \right) = 0.$$

As explained in the Introduction we shall consider

$$\psi(t, x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} (a + \bar{u}) \left(\frac{1}{t}, \frac{x}{t} \right).$$

We define

$$\alpha(t, x) = \Re \psi(t, x) \quad , \quad \beta(t, x) = \Im \psi(t, x),$$

and then an orthonormal frame (T, e_1, e_2) by imposing

$$\begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}_x (t, x) = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix} (t, x) \quad ,$$

and

$$\begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}_t(t, 0) = \begin{pmatrix} 0 & -\beta_x & \alpha_x \\ \beta_x & 0 & \gamma \\ -\alpha_x & -\gamma & 0 \end{pmatrix} \begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}(t, 0),$$

with γ to be specified later. For all (t, x) we denote by $(a, b, c)(t, x)$ the functions such that

$$\begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}_t(t, x) = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}(t, x).$$

We first notice that $(a, b, c)(t, 0) = (-\beta_x, \alpha_x, \gamma)(t, 0)$. By computing

$$T_{tx} = a_x e_1 + b_x e_2 - (a\alpha + b\beta)T, \quad T_{xt} = \alpha_t e_1 + \beta_t e_2 + \alpha e_{1t} + \beta e_{2t},$$

$$e_{1tx} = -a_x T + c_x e_2 - a(\alpha e_1 + \beta e_2) - c\beta T, \quad e_{1xt} = -\alpha_t T - \alpha(ae_1 + be_2),$$

we obtain that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & -\beta \\ 0 & 0 & \alpha \\ \beta & -\alpha & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} \alpha_t \\ \beta_t \\ 0 \end{pmatrix},$$

which is equivalent to

$$\begin{cases} i(\alpha + i\beta)_t + (b - ia)_x - c(\alpha + i\beta) = 0, \\ c_x = -\left(\frac{\alpha^2 + \beta^2}{2}\right)_x, \end{cases},$$

so we obtain $(a, b, c)(t, x) = (-\beta_x, \alpha_x, \gamma)(t, x)$ with $\gamma(t, x) = -\frac{|\psi|^2(t, x)}{2} + \frac{a^2}{2t}$.

Define $\chi(t, x)$ as

$$\chi(t, x) = \chi(t_0, x_0) + \int_0^t (T \wedge T_{xx})(t', x_0) dt' + \int_{x_0}^x T(t, s) ds.$$

Since T is a solution of

$$T_t = -\beta_x e_1 + \alpha_x e_2 = T \wedge T_{xx},$$

we conclude that χ solves the binormal flow (1).

In conclusion, given a solution of the cubic 1-D NLS (16), we can construct an orthonormal frame (T, e_1, e_2) which leads to a solution of the binormal flow (1). Finally we compute the derivatives of the tangent vector and of the normal complex vector $N = e_1 + ie_2$ in terms of ψ . This will be useful in the following sections: ³

$$(17) \quad T_x = \alpha e_1 + \beta e_2 = \Re \bar{\psi} N,$$

$$(18) \quad N_x = e_{1x} + ie_{2x} = -\alpha T - i\beta T = -\psi T,$$

$$(19) \quad T_t = \alpha_x e_2 - \beta_x e_1 = \Im \bar{\psi}_x N,$$

³we also have $iT_t + T_{xx} = \bar{\psi}_x N + \Re \bar{u} N_x = \bar{\psi}_x N - |\psi|^2 T$, but we will not use it.

$$(20) \quad N_t = \beta_x T + \gamma e_2 - i\alpha_x T - i\gamma e_1 = -i\psi_x T - i\gamma N.$$

Remark 2.1. In the case of the Frenet frame one defines c and τ from ψ by

$$c(t, x) = |\psi(t, x)|, \quad \tau(t, x) = \Im \frac{\psi_x(t, x)}{\psi(t, x)},$$

then the frame (T, n, b) by

$$\begin{pmatrix} T \\ n \\ b \end{pmatrix}_x = \begin{pmatrix} 0 & c & 0 \\ -c & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ n \\ b \end{pmatrix}, \quad \begin{pmatrix} T \\ n \\ b \end{pmatrix}_t = \begin{pmatrix} 0 & -c\tau & c_x \\ c\tau & 0 & \frac{c_{xx} - c\tau^2}{c} \\ -c_x & -\frac{c_{xx} - c\tau^2}{c} & 0 \end{pmatrix} \begin{pmatrix} T \\ n \\ b \end{pmatrix}.$$

One can see the link between these two constructions by considering (see [19] and also page 5 of [16])

$$\begin{aligned} e_1(t, x) &= \cos \int_0^x \tau(t, s) ds n(t, x) - \sin \int_0^x \tau(t, s) ds b(t, x), \quad e_1(0, x) = n(0, x), \\ e_2(t, x) &= \sin \int_0^x \tau(t, s) ds n(t, x) + \cos \int_0^x \tau(t, s) ds b(t, x), \quad e_2(0, x) = b(0, x), \end{aligned}$$

so

$$\begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}_x = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix},$$

with

$$\alpha(t, x) = c(t, x) \cos \int_0^x \tau(t, s) ds, \quad \beta(t, x) = c(t, x) \sin \int_0^x \tau(t, s) ds.$$

Moreover, one gets that the complex normal vector N is written as

$$N = e_1 + ie_2 = n(\cos + i \sin) + b(-\sin + i \cos) = (n + ib)(\cos + i \sin) = (n + ib)e^{i \int_0^x \tau(t, s) ds}.$$

3. ASYMPTOTICS IN SPACE FOR THE TANGENT VECTOR AND THE NORMAL VECTORS

3.1. The limit in space for $T(t, x)$ for a fixed given t . Recall that $T(t, x)$ solves (17) and that $\alpha(t, x) + i\beta(t, x) = \psi(t, x)$ with

$$\psi(t, x) = \frac{e^{i \frac{x^2}{4t}}}{\sqrt{t}} (a + \bar{u}) \left(\frac{1}{t}, \frac{x}{t} \right).$$

In what follows we are going to make a repeated use of integration by parts trying to exploit the high oscillations of the function $\frac{e^{i \frac{x^2}{4t}}}{\sqrt{t}}$. From (17) we get

$$\int_x^\infty T_s(t, s) ds = \Re \int_x^\infty \bar{\psi} N(t, s) ds = \Re \int_x^\infty \frac{e^{-i \frac{s^2}{4t}}}{\sqrt{t}} (a + u) \left(\frac{1}{t}, \frac{s}{t} \right) N(t, s) ds$$

$$\begin{aligned}
&= -\Re \frac{2t}{-ix} \bar{\psi}(t, x) N(t, x) - \Re \int_x^\infty \frac{2t}{is^2} \bar{\psi}(t, s) N(t, s) ds \\
&\quad - \Re \int_x^\infty e^{-i\frac{s^2}{4t}} \frac{2}{-is\sqrt{t}} (u_s) \left(\frac{1}{t}, \frac{s}{t} \right) N(t, s) ds - \Re \int_x^\infty \frac{2t}{-is} \bar{\psi}(t, s) N_s(t, s) ds.
\end{aligned}$$

First we notice that in view of (18) the last integral is purely imaginary so the last term vanishes. Then, since $u \in L_{tx}^\infty$ and $|N| = 2$ it follows that

$$(21) \quad \left| \int_x^\infty T_s(t, s) ds - \Im \int_x^\infty e^{-i\frac{s^2}{4t}} \frac{2}{s\sqrt{t}} (u_s) \left(\frac{1}{t}, \frac{s}{t} \right) N(t, s) ds \right| \leq C(u_1) \frac{\sqrt{t}}{x}.$$

This implies by Cauchy-Schwarz that there is a limit $T^\infty(t)$ for $T(t, x)$ as x goes to infinity and

$$(22) \quad |T(t, x) - T^\infty(t)| \leq C(u_1) \left(\frac{1}{\sqrt{x}} + \frac{\sqrt{t}}{x} \right).$$

3.2. $T^\infty(t)$ is independent of time. Let $0 < t \leq 1$, $1 < x$ and recall (19). Let $0 < \epsilon$. We have

$$\begin{aligned}
T(t, x) - T(1, x) &= \int_t^1 T_{t'}(t', x) dt' = \Im \int_t^1 \bar{\psi}_x(t', x) N(t', x) dt' \\
&= \Im \int_t^1 \frac{e^{-i\frac{x^2}{4t'}}}{2t'\sqrt{t'}} (2(u_x) - ix(a+u)) \left(\frac{1}{t'}, \frac{x}{t'} \right) N(t', x) dt'.
\end{aligned}$$

Again we will exploit the high oscillations of the function $\frac{e^{i\frac{x^2}{4t'}}}{\sqrt{t'}}$, by integrating by parts

$$\begin{aligned}
T(t, x) - T(1, x) &= \left[\Im e^{-i\frac{x^2}{4t'}} \frac{2\sqrt{t'}}{ix^2} (2(u_x) - ix(a+u)) \left(\frac{1}{t'}, \frac{x}{t'} \right) N(t', x) \right]_t^1 \\
&\quad - \Im \int_t^1 e^{-i\frac{x^2}{4t'}} \partial_{t'} \left(\frac{2\sqrt{t'}}{ix^2} (2(u_x) - ix(a+u)) \left(\frac{1}{t'}, \frac{x}{t'} \right) N(t', x) \right) dt'.
\end{aligned}$$

Using the fact that u and its derivative are bounded, we obtain the existence of a constant $C(u_1)$, depending only on $\|u_1\|_{X_1^\gamma}$ and t such that

$$\begin{aligned}
|T(t, x) - T(1, x)| &\leq \frac{C(u_1, t)}{x} \\
&\quad + \left| \Im \int_t^1 e^{-i\frac{x^2}{4t'}} \frac{2\sqrt{t'}}{ix^2} \partial_{t'} (2(u_x) - ixu) \left(\frac{1}{t'}, \frac{x}{t'} \right) N(t', x) dt' \right| \\
&\quad + \left| \Im \int_t^1 e^{-i\frac{x^2}{4t'}} \frac{2\sqrt{t'}}{ix^2} (2(u_x) - ix(a+u)) \left(\frac{1}{t'}, \frac{x}{t'} \right) (-i\psi_x T - i\gamma N)(t', x) dt' \right|.
\end{aligned}$$

In the last integral we have used the expression (20) of $\partial_{t'} N$. Let us notice that u_t , u_x , u_{tx} and u_{xx} are in L_{tx}^∞ because we are assuming that $u_1 \in H^4$ and we shall include in $C(u_1, t)$ this dependence. Therefore the first integral is upper-bounded by $\frac{C(u_1, t)}{x}$, except for the

term where the derivative in time falls on u and we loose the inverse powers of x . We left this term aside for the moment. Since $\gamma(t, x) = -\frac{|\psi|^2(t, x)}{2} + \frac{a^2}{2t}$,

$$|i\gamma N(t', x)| \leq \frac{C(u_1)}{t'},$$

so the contribution of this term is of order $\frac{C(u_1, t)}{x}$. Concerning the $-i\psi_x T$ remaining part, recall that

$$\psi_x(t', x) = \frac{e^{i\frac{x^2}{4t'}}}{2t'\sqrt{t'}}(2(\bar{u}_x) + ix(a + \bar{u})) \left(\frac{1}{t'}, \frac{x}{t'} \right),$$

so we get again a $\frac{C(u_1, t)}{x}$ bound except for the term with no inverse power of x . But the integrant of this term is real, so the term is null. Summarizing, we have

$$|T(t, x) - T(1, x)| \leq \frac{C(u_1, t)}{x} + \left| \Im \int_t^1 e^{-i\frac{x^2}{4t'}} \frac{2}{t'\sqrt{t'}} (u_x) \left(\frac{1}{t'}, \frac{x}{t'} \right) N(t', x) dt' \right|.$$

The integral is of the same type as the first term in the initial expression of $T(t, x) - T(1, x)$, and, as we have seen above, it can be upper-bounded by $\frac{C(u_1, t)}{x}$. Therefore

$$|T(t, x) - T(1, x)| \leq \frac{C(u_1, t)}{x},$$

with $C(u_1, t)$ is depending on $\|u_1\|_{X_1^\gamma}$, $\|u_1\|_{H^4}$ and t . By taking x large with respect to t^{-1} and to ϵ^{-1} and by using also (22) we obtain that

$$|T(t, x) - T(1, x)| \leq \epsilon, \quad |T(1, x) - T^\infty(1)| \leq \epsilon, \quad |T(t, x) - T^\infty(t)| \leq \epsilon,$$

so

$$T^\infty(t) = T^\infty(1) = T^\infty.$$

Therefore, in view of (22), the first part of (i) in Theorem 1.1 is proved. Moreover, (21) becomes

$$(23) \quad \left| T(t, x) - T^\infty + \Im \int_x^\infty \frac{2}{s\sqrt{t}} (u_s) \left(\frac{1}{t}, \frac{s}{t} \right) e^{-i\frac{s^2}{4t}} N(t, s) ds \right| \leq C(u_1) \frac{\sqrt{t}}{x}.$$

3.3. The limit in space for $N(t, s)$ for a fixed t . We define

$$\tilde{N}(t, x) = N(t, x)e^{i\Phi}, \quad \Phi(t, x) = -\frac{a^2}{2} \log t + a^2 \log x.$$

Integrating by parts again we get

$$\begin{aligned} \int_x^\infty \tilde{N}_s(t, s) ds &= \int_x^\infty \left(-\psi T + \frac{ia^2}{s} N \right) e^{i\Phi} \\ &= \frac{2t}{ix} \psi T e^{i\Phi} - \int_x^\infty \frac{2t}{is^2} \psi T e^{i\Phi} ds \\ &+ \int_x^\infty e^{i\frac{s^2}{4t}} \frac{2}{is\sqrt{t}} (\bar{u}_s) \left(\frac{1}{t}, \frac{s}{t} \right) T(t, s) e^{i\Phi} ds + \int_x^\infty e^{i\frac{s^2}{4t}} \frac{2\sqrt{t}}{is} (a + \bar{u}) \left(\frac{1}{t}, \frac{s}{t} \right) T_s(t, s) e^{i\Phi} ds \end{aligned}$$

$$-a^2 \int_x^\infty \frac{2t}{s^2} \psi T e^{i\Phi} ds + \int_x^\infty \frac{ia^2}{s} N e^{i\Phi}.$$

In view of the $\frac{C(u_1)}{\sqrt{t}}$ bound on ψ , the first two terms and the fifth one are upper-bounded by $\frac{C(u_1)\sqrt{t}}{x}$. Formula (17) insures us that $T_s = \Re \bar{\psi} N$, and by using Cauchy-Schwarz inequality we can upper-bound by $\frac{C(u_1)\sqrt{t}}{\sqrt{x}}$ the part involving u in the fourth term. We get

$$\begin{aligned} & \left| \int_x^\infty \tilde{N}_s(t, s) ds - \int_x^\infty e^{i\frac{s^2}{4t}} \frac{2}{is\sqrt{t}} (\bar{u}_s) \left(\frac{1}{t}, \frac{s}{t} \right) T(t, s) e^{i\Phi} ds \right| \\ & \leq C(u_1) \left(\frac{\sqrt{t}}{x} + \frac{\sqrt{t}}{\sqrt{x}} \right) + \left| \int_x^\infty e^{i\frac{s^2}{4t}} \frac{2\sqrt{t}}{is} a \Re \left(\frac{ae^{-i\frac{s^2}{4t}}}{\sqrt{t}} N \right) e^{i\Phi} ds + \int_x^\infty \frac{ia^2}{s} N e^{i\Phi} \right|. \end{aligned}$$

We obtain then the cancelation of the non-oscillatory terms involving N ,

$$\begin{aligned} & \left| \int_x^\infty \tilde{N}_s(t, s) ds - \int_x^\infty e^{i\frac{s^2}{4t}} \frac{2}{is\sqrt{t}} (\bar{u}_s) \left(\frac{1}{t}, \frac{s}{t} \right) T(t, s) e^{i\Phi} ds \right| \\ & \leq C(u_1) \left(\frac{\sqrt{t}}{x} + \frac{\sqrt{t}}{\sqrt{x}} \right) + \left| \int_x^\infty e^{i\frac{s^2}{4t}} \frac{a^2}{is} \bar{N} e^{i\Phi} ds \right|. \end{aligned}$$

By performing a last integration by parts we have

$$\int_x^\infty \frac{e^{i\frac{s^2}{4t}}}{s} \bar{N} e^{i\Phi} ds = -e^{i\frac{x^2}{4t}} \frac{2t}{ix^2} \bar{N} e^{i\Phi} - \int_x^\infty e^{i\frac{s^2}{4t}} \frac{2t}{i} \left(-\frac{2}{s^3} \bar{N} + \frac{ia^2}{s^3} \bar{N} + \frac{1}{s^2} \bar{N}_s \right) e^{i\Phi}.$$

From (18) we have $N_s = -\psi T$, so we get an upper bound of $|N_s|$ of the type $\frac{C}{\sqrt{t}}$. Hence we finally obtain

$$(24) \quad \left| \int_x^\infty \tilde{N}_s(t, s) ds - \int_x^\infty e^{i\frac{s^2}{4t}} \frac{2}{is\sqrt{t}} (\bar{u}_s) \left(\frac{1}{t}, \frac{s}{t} \right) T(t, s) e^{i\Phi} ds \right| \leq C(u_1) \left(\frac{\sqrt{t}}{x} + \frac{t}{x^2} + \frac{\sqrt{t}}{\sqrt{x}} \right).$$

By Cauchy-Schwarz inequality we deduce that $\tilde{N}(t, x)$ has a limit $N^\infty(t)$ as x goes to infinity and

$$(25) \quad \left| \tilde{N}(t, x) - N^\infty(t) \right| \leq C(u_1) \left(\frac{1}{\sqrt{x}} + \frac{\sqrt{t}}{x} + \frac{t}{x^2} + \frac{\sqrt{t}}{\sqrt{x}} \right).$$

3.4. $N^\infty(t)$ is independent of time. Let $0 < t \leq 1$, $1 < x$. Let $0 < \epsilon$. We have in view of (20)

$$\begin{aligned} \int_t^1 \tilde{N}_{t'}(t', x) dt' &= \int_t^1 \left(N_{t'} - \frac{ia^2}{2t'} N \right) e^{i\Phi} dt' = \int_t^1 \left(-i\psi_x T - i\gamma N - \frac{ia^2}{2t'} N \right) e^{i\Phi} dt' \\ &= \int_t^1 \left((-i) \frac{e^{i\frac{x^2}{4t'}}}{2t'\sqrt{t'}} (2(\bar{u}_x) + ix(a + \bar{u})) \left(\frac{1}{t'}, \frac{x}{t'} \right) T - i\gamma N - \frac{ia^2}{2t'} N \right) e^{i\Phi} dt'. \end{aligned}$$

As in §3.2, in the term involving T we perform integrations by parts in time relying on the oscillations of $e^{i\frac{x^2}{4t}}$ to obtain

$$\begin{aligned} |\tilde{N}(t', x) - \tilde{N}(1, x)| &\leq \frac{C(u_1, t)}{x} + \left| \int_t^1 \left(-i\gamma N - \frac{ia^2}{2t'} N \right) e^{i\Phi} dt' \right. \\ &\quad \left. + \int_t^1 e^{i\frac{x^2}{4t'}} \left(\frac{2i}{t'\sqrt{t'}} (\overline{u}_x) \left(\frac{1}{t'}, \frac{x}{t'} \right) T - \frac{2\sqrt{t'}}{x^2} (2(\overline{u}_x) + ix(a + \overline{u})) \left(\frac{1}{t'}, \frac{x}{t'} \right) T_{t'} \right) e^{i\Phi} dt' \right|. \end{aligned}$$

In the integral involving T we perform again an integration by parts in time, and we use expression (19): $T_{t'} = \Im \overline{\psi}_x N$. A $\frac{C(u_1, t)}{x}$ upper-bound follows also for this part. From the last term we keep only the one without an inverse power of x , corresponding to the part of $\overline{\psi}_x$ where the derivative falls on the phasis. We have then

$$\begin{aligned} |\tilde{N}(t', x) - \tilde{N}(1, x)| &\leq \frac{C(u_1, t)}{x} + \left| \int_t^1 \left(-i\gamma N - \frac{ia^2}{2t'} N \right) e^{i\Phi} dt' \right. \\ &\quad \left. - \int_t^1 e^{i\frac{x^2}{4t'}} 2\sqrt{t'} i(a + \overline{u}) \left(\frac{1}{t'}, \frac{x}{t'} \right) \Im \left(\frac{e^{-i\frac{x^2}{4t'}}}{2t'\sqrt{t'}} (-i(a + u)) \left(\frac{1}{t'}, \frac{x}{t'} \right) N \right) e^{i\Phi} dt' \right|. \end{aligned}$$

We recall that $\gamma = -\frac{|\psi|^2}{2} + \frac{a^2}{2t}$ only involves powers of $u \left(\frac{1}{t'}, \frac{x}{t'} \right)$, so

$$\begin{aligned} |\tilde{N}(t', x) - \tilde{N}(1, x)| &\leq \frac{C(u_1, t)}{x} + C \int_t^1 \left| u \left(\frac{1}{t'}, \frac{x}{t'} \right) \right| \frac{dt'}{t'} \\ &\quad \left| - \int_t^1 \frac{ia^2}{2t'} N e^{i\Phi} dt' + \int_t^1 e^{i\frac{x^2}{4t'}} \frac{ia^2}{t'} \Re \left(e^{-i\frac{x^2}{4t'}} N \right) e^{i\Phi} dt' \right| \\ &\leq \frac{C(u_1, t)}{x} + C \int_t^1 \left| u \left(\frac{1}{t'}, \frac{x}{t'} \right) \right| \frac{dt'}{t'} + \left| \int_t^1 e^{i\frac{x^2}{2t'}} \frac{ia^2}{2t'} \overline{N} e^{i\Phi} dt' \right|. \end{aligned}$$

We perform a last integration by parts of the oscillating function $e^{i\frac{x^2}{4t}}$ in the \overline{N} -term and use formula (20): $N_t = -i\psi_x T - i\gamma N$. This way we obtain that this term has also the desired decay $\frac{C(u_1, t)}{x}$. In conclusion

$$|\tilde{N}(t', x) - \tilde{N}(1, x)| \leq \frac{C(u_1, t)}{x} + C \int_t^1 \left| u \left(\frac{1}{t'}, \frac{x}{t'} \right) \right| \frac{dt'}{t'}.$$

Like in Lemma B.1 of [4] we can show that if $xu(1)$ and $x\partial_x u$ are in L^2 then this regularities are preserved, and the L^2 norms of $xu(t)$ and $x\partial_x u(t)$ are controlled by some polynomial growth in time. In particular we can estimate

$$\int_t^1 \left| u \left(\frac{1}{t'}, \frac{x}{t'} \right) \right| \frac{dt'}{t'} \leq \frac{1}{x} \int_t^1 \frac{x}{t'} \left| u \left(\frac{1}{t'}, \frac{x}{t'} \right) \right| dt' \leq \frac{C}{x} \|xu_1\|_{L^2}^{\frac{1}{2}} \|x\partial_x u_1\|_{L^2}^{\frac{1}{2}}.$$

Therefore

$$|\tilde{N}(t', x) - \tilde{N}(1, x)| \leq \frac{C(u_1, t)}{x},$$

and as in §2.2 we conclude that $\tilde{N}^\infty(t) = \tilde{N}^\infty(1) = N^\infty$. In particular, (25) writes

$$(26) \quad \left| \tilde{N}(t, x) - N^\infty \right| \leq C(u_1) \left(\frac{1}{\sqrt{x}} + \frac{\sqrt{t}}{x} + \frac{t}{x^2} \right),$$

and (24) becomes

$$(27) \quad \left| \tilde{N}(t, x) - N^\infty - i \int_x^\infty e^{i \frac{s^2}{4t}} \frac{2}{s\sqrt{t}} (\bar{u}_s) \left(\frac{1}{t}, \frac{s}{t} \right) T(t, s) e^{i\Phi} ds \right| \leq C(u_1) \left(\frac{\sqrt{t}}{x} + \frac{t}{x^2} + \sqrt{t} \right).$$

Finally, recall that

$$\tilde{N}(t, x) = N(t, x) e^{i\Phi(t, x)} = N(t, x) e^{-ia^2 \log \frac{\sqrt{t}}{x}}$$

and we obtain from (26) the second part of (i) in Theorem 1.1.

4. THE LIMIT OF $T(t, x)$ AS t GOES TO 0

Let $x > 0$. Combining (23) and (27)

$$(28) \quad \left| T(t, x) - T^\infty + \Im N^\infty \int_x^\infty h(t, s) ds + \Re \int_x^\infty h(t, s) \int_s^\infty \overline{h(t, s')} T(t, s') ds' ds \right| \leq C(u_1) \left(\frac{\sqrt{t}}{x} + \frac{t}{x^2} + \sqrt{t} \right),$$

where

$$h(t, s) = e^{-i \frac{s^2}{4t}} \frac{2}{s\sqrt{t}} (u_s) \left(\frac{1}{t}, \frac{s}{t} \right) e^{-i\Phi}.$$

In the next subsection we prove two estimates that will allow us to handle the integrals in (28).

4.1. Two integral estimates.

Lemma 4.1. *There exists $C > 0$ such that for all t small with respect to u_1 and x , we have*

$$\int_x^\infty |h(t, s)| ds \leq C(u_1).$$

Proof. On the one hand, by Cauchy-Schwarz inequality, if $x \geq 1$,

$$\int_1^\infty \frac{2}{s\sqrt{t}} \left| (u_s) \left(\frac{1}{t}, \frac{s}{t} \right) \right| ds \leq C(u_1).$$

On the other hand, if $x \leq 1$, we shall introduce the J operator (see Appendix 6)

$$\begin{aligned} \int_x^1 \frac{2}{s\sqrt{t}} \left| (u_s) \left(\frac{1}{t}, \frac{s}{t} \right) \right| ds &= \int_{\frac{x}{t}}^{\frac{1}{t}} \frac{2\sqrt{t}}{s} \left| (u_s) \left(\frac{1}{t}, s \right) \right| ds \\ &\leq \int_{\frac{x}{t}}^{\frac{1}{t}} \frac{4\sqrt{t}}{s} \left(\left| (Ju) \left(\frac{1}{t}, s \right) \right| + \left| su \left(\frac{1}{t}, s \right) \right| \right) ds, \end{aligned}$$

so by Cauchy-Schwarz inequality,

$$\begin{aligned} \int_x^1 \frac{2}{s\sqrt{t}} \left| (u_s) \left(\frac{1}{t}, \frac{s}{t} \right) \right| ds &\leq C \frac{t}{x} \left\| Ju \left(\frac{1}{t} \right) \right\|_{L^2} + C\sqrt{t} \sqrt{\frac{1-x}{t}} \left\| u \left(\frac{1}{t} \right) \right\|_{L^2} \\ &\leq C \frac{t}{x} \left\| Ju \left(\frac{1}{t} \right) \right\|_{L^2} + C \|u_1\|_{L^2}. \end{aligned}$$

In Proposition 6.3 we prove that

$$\left\| Ju \left(\frac{1}{t} \right) \right\|_{L^2} \leq \frac{C(u_1)}{t^{\frac{3}{4}}},$$

so the Lemma follows. \square

Remark 4.2. Combining (23) with Lemma 4.1 we obtain that for all $x > 0$ and t small with respect to u_1 and x ,

$$|T(t, x) - T^\infty| \leq C(u_1),$$

and the first part of (ii) in Theorem 1.1 follows.

From (22) applied to $t = 1$ and $x > 1$ we obtain

$$|T(1, x) - T^\infty| \leq C(u_1),$$

From the results in [4] we have $|T(1, x) - T_a(1, x)| \leq C(u_1)$. We can use this estimate in our context of less decay in time for the convergence to the asymptotic state because we are considering only the value $t = 1$. Therefore the angle between T^∞ and $T^{-\infty}$ is $C(u_1)$ -close to the selfsimilar one, and we obtain the (ii) part of Theorem 1.1.

In [5] we have obtained $|\chi(t, x) - \chi(0, x)| \leq C(u_1)\sqrt{t}$. As a consequence, for $x, \tilde{x} > 0$ and t small with respect to u_1 and to x and \tilde{x} , we get

$$\begin{aligned} |\chi(0, x) - \chi(0, \tilde{x}) - T^\infty(x - \tilde{x})| &\leq C(u_1)\sqrt{t} + |\chi(t, x) - \chi(t, \tilde{x}) - T^\infty(x - \tilde{x})| \\ &\leq C(u_1)\sqrt{t} + \left| \int_{\tilde{x}}^x T(t, s) - T^\infty ds \right| \leq C(u_1)\sqrt{t} + C(u_1)(x - \tilde{x}) \leq C(u_1)(x - \tilde{x}), \end{aligned}$$

so the part (iii) of Theorem 1.1 also follows.

Lemma 4.3. For all $g \in L^\infty$ with $g_s \in L^1 \cap L^2$, $0 < x \leq \tilde{x}$,

$$\left| \int_x^{\tilde{x}} h(t, s) g(s) ds - i \int_x^{\tilde{x}} \widehat{f}_+ \left(\frac{s}{2} \right) g(s) \frac{ds}{s^{ia^2}} \right| \leq C(g, u_1) \left(\frac{\sqrt{t}}{x} + t^{\frac{1}{4}-} \right).$$

Proof. We obtain by the scattering result

$$\begin{aligned} \int_x^{\tilde{x}} \frac{1}{s\sqrt{t}} \left((u_s) \left(\frac{1}{t}, \frac{s}{t} \right) - \sqrt{t} e^{-i\frac{a^2}{2} \log t} e^{i\frac{s^2}{4t}} \int e^{i\frac{y^2 t}{4}} e^{-i\frac{sy}{2}} \partial_y f_+(y) dy \right) e^{-i\frac{s^2}{4t}} e^{-i\Phi} g(s) ds \\ = \int_x^{\tilde{x}} \frac{1}{s\sqrt{t}} (r_s) \left(\frac{1}{t}, \frac{s}{t} \right) e^{-i\frac{s^2}{4t}} e^{-i\Phi} g(s) ds, \end{aligned}$$

with

$$\left\| r \left(\frac{1}{t} \right) \right\|_{H^2} \leq C(u_1) t^{\frac{1}{4}-}.$$

We shall first show that this remainder term can be upper-bounded as in the statement of the Lemma. For $x \geq 1$ by Cauchy-Schwarz,

$$\left| \int_x^{\tilde{x}} \frac{1}{s\sqrt{t}} (r_s) \left(\frac{1}{t}, \frac{s}{t} \right) e^{-i\frac{s^2}{4t}} e^{-i\Phi} g(s) ds \right| \leq C(g, u_1) t^{\frac{1}{4}-}.$$

Then for $x \geq 1$ we need to treat only the case $\tilde{x} = 1$, and we shall do this by integrating by parts

$$\begin{aligned} \int_x^1 \frac{1}{s\sqrt{t}} (r_s) \left(\frac{1}{t}, \frac{s}{t} \right) e^{-i\frac{s^2}{4t}} e^{-i\Phi} g(s) ds &= \left[\frac{\sqrt{t}}{s} r \left(\frac{1}{t}, \frac{s}{t} \right) e^{-i\frac{s^2}{4t}} e^{-i\Phi} g(s) \right]_x^1 \\ &+ \int_x^1 \frac{\sqrt{t}}{s^2} r \left(\frac{1}{t}, \frac{s}{t} \right) e^{-i\frac{s^2}{4t}} e^{-i\Phi} g(s) ds + \int_x^1 \frac{\sqrt{t}}{s} r \left(\frac{1}{t}, \frac{s}{t} \right) \frac{is}{2t} e^{-i\frac{s^2}{4t}} e^{-i\Phi} g(s) ds \\ &+ \int_x^1 \frac{\sqrt{t}}{s} r \left(\frac{1}{t}, \frac{s}{t} \right) e^{-i\frac{s^2}{4t}} \frac{ia^2}{s} e^{-i\Phi} g(s) ds - \int_x^1 \frac{\sqrt{t}}{s} r \left(\frac{1}{t}, \frac{s}{t} \right) e^{-i\frac{s^2}{4t}} e^{-i\Phi} g_s(s) ds. \end{aligned}$$

By a simple integration and Cauchy-Schwarz we obtain

$$\begin{aligned} \left| \int_x^1 \frac{1}{s\sqrt{t}} (r_s) \left(\frac{1}{t}, \frac{s}{t} \right) e^{-i\frac{s^2}{4t}} e^{-i\Phi} g(s) ds \right| &\leq C \frac{\sqrt{t}}{x} \left\| r \left(\frac{1}{t} \right) \right\|_{L^\infty} \|g\|_{L^\infty} \\ &+ C \left\| r \left(\frac{1}{t} \right) \right\|_{L^2} \|g\|_{L^\infty} + C \frac{t}{x} \left\| r \left(\frac{1}{t} \right) \right\|_{L^2} \|g_s\|_{L^2} \leq C(g, u_1) t^{\frac{1}{4}-} + C(g, u_1) \frac{\sqrt{t}}{x}. \end{aligned}$$

In conclusion

$$\begin{aligned} \left| \int_x^{\tilde{x}} h(t, s) g(s) ds - \int_x^{\tilde{x}} \int e^{i\frac{y^2 t}{4}} e^{-i\frac{sy}{2}} \partial_y f_+(y) dy \frac{2g(s)}{s^{1+ia^2}} ds \right| \\ \leq C(g, u_1) t^{\frac{1}{4}-} + C(g, u_1) \frac{\sqrt{t}}{x}. \end{aligned}$$

Since

$$\int e^{i\frac{y^2 t}{4}} e^{-i\frac{sy}{2}} \partial_y f_+(y) dy = i\frac{s}{2} \widehat{f_+} \left(\frac{s}{2} \right) + \int \left(e^{i\frac{y^2 t}{4}} - 1 \right) e^{-i\frac{sy}{2}} \partial_y f_+(y) dy,$$

it follows that in order to obtain the Lemma it is enough to estimate

$$\begin{aligned} \int_x^{\tilde{x}} \int \left(e^{i\frac{y^2 t}{4}} - 1 \right) e^{-i\frac{sy}{2}} \partial_y f_+(y) dy \frac{2g(s)}{s^{1+ia^2}} ds \\ = \int \mathcal{F} \left(\left(e^{i\frac{y^2 t}{4}} - 1 \right) \partial_y f_+(y) \right) \left(\frac{s}{2} \right) \frac{2g(s) \mathbb{I}_{(x, \tilde{x})}(s)}{s^{1+ia^2}} ds \\ = \int \left(e^{i\frac{y^2 t}{4}} - 1 \right) \partial_y f_+(y) \int_x^{\tilde{x}} \frac{e^{-i\frac{sy}{2}} g(s)}{s^{1+ia^2}} ds dy = I(x, \tilde{x}). \end{aligned}$$

In the last equality we have used Parseval identity. We shall need some estimates for $y \neq 0$. One has

$$\int_x^{\tilde{x}} \frac{e^{-i\frac{sy}{2}} g(s)}{s^{1+ia^2}} ds = \frac{2e^{-i\frac{s\tilde{y}}{2}} g(s)}{-iy s^{1+ia^2}} \Big|_x^{\tilde{x}} - \int_x^{\tilde{x}} \frac{2e^{-i\frac{sy}{2}}}{-iy} \left(-\frac{(1+ia^2)g(s)}{s^{2+ia^2}} + \frac{g(s)}{s^{1+ia^2}} \right) ds,$$

so

$$(29) \quad \left| \int_x^{\tilde{x}} \frac{e^{-i\frac{sy}{2}} g(s)}{s^{1+ia^2}} ds \right| \leq C \frac{\|g\|_{L^\infty}}{|xy|} + C \frac{\|g_s\|_{L^2}}{|y|\sqrt{x}} \leq C(g) \frac{1}{|y|} \left(\frac{1}{x} + 1 \right),$$

Also for all $\alpha > 0$ and $|y| \geq 1$,

$$(30) \quad \left| \int_x^1 \frac{e^{-i\frac{sy}{2}} g(s)}{s^{ia^2}} ds \right| \leq C(g, \alpha) \frac{1}{|x|^\alpha |y|}.$$

Indeed, by integrating by parts

$$\left| \int_x^1 \frac{e^{-i\frac{sy}{2}} g(s)}{s^{ia^2}} ds \right| \leq C \left(\frac{\|g\|_{L^\infty}}{|y|} + a^2 \frac{\log x \|g\|_{L^\infty}}{|y|} + \frac{\|g_s\|_{L^1}}{|y|} \right) \leq C(g, \alpha) \frac{1}{|x|^\alpha |y|}.$$

It is enough to treat $I(x, 1)$ and $I(1, \tilde{x})$ for all $0 < x \leq 1 \leq \tilde{x}$. From (29) and by Cauchy-Schwarz inequality we get

$$\begin{aligned} |I(1, \tilde{x})| &\leq C(g) \left(\int_{|y| \leq \frac{1}{\sqrt{t}}} \left| \frac{e^{i\frac{y^2 t}{4}} - 1}{y} \right| |\partial_y f_+(y)| dy + \int_{|y| \geq \frac{1}{\sqrt{t}}} \left| e^{i\frac{y^2 t}{4}} - 1 \right| \left| \frac{\partial_y f_+(y)}{y} \right| dy \right) \\ &\leq C(g) \left(\int_{|y| \leq \frac{1}{\sqrt{t}}} t |y| |\partial_y f_+(y)| dy + \int_{|y| \geq \frac{1}{\sqrt{t}}} \left| \frac{\partial_y f_+(y)}{y} \right| dy \right) \leq C(g, u_1) t^{\frac{1}{4}}. \end{aligned}$$

For treating $I(x, 1)$ we need to introduce a cutoff function $\eta(t|y|)$ such that $\eta(r) = 1$ for $|r| \leq 1$ and $\eta(r) = 0$ for $|r| \geq 2$. On one hand by (29) and by Cauchy-Schwarz inequality

$$\begin{aligned} &\left| \int \left(e^{i\frac{y^2 t}{4}} - 1 \right) \partial_y f_+(y) (1 - \eta(t|y|)) \int_x^1 \frac{e^{-i\frac{sy}{2}} g(s)}{s^{1+ia^2}} ds dy \right| \\ &\leq \int_{\frac{1}{t} \leq |y|} \frac{C(g)}{x|y|} |\partial_y f_+(y)| dy \leq C(g, u_1) \frac{\sqrt{t}}{x}. \end{aligned}$$

On the remaining part of $I(x, 1)$ we shall perform an integration by parts

$$\begin{aligned} &\int \left(e^{i\frac{y^2 t}{4}} - 1 \right) \partial_y f_+(y) \eta(t|y|) \int_x^1 \frac{e^{-i\frac{sy}{2}} g(s)}{s^{1+ia^2}} ds dy \\ &= - \int \frac{iyt}{2} e^{i\frac{y^2 t}{4}} f_+(y) \eta(t|y|) \int_x^1 \frac{e^{-i\frac{sy}{2}} g(s)}{s^{1+ia^2}} ds dy \\ &\quad - \int \left(e^{i\frac{y^2 t}{4}} - 1 \right) f_+(y) \eta_y(t|y|) \int_x^1 \frac{e^{-i\frac{sy}{2}} g(s)}{s^{1+ia^2}} ds dy \end{aligned}$$

$$+ \int \left(e^{i\frac{y^2 t}{4}} - 1 \right) f_+(y) \eta(t|y|) \int_x^1 \frac{ie^{-i\frac{sy}{2}} g(s)}{2s^{ia^2}} ds dy = I_1 + I_2 + I_3.$$

For I_1 and I_2 we use (29) and Cauchy-Schwarz inequality

$$|I_1 + I_2| \leq \int_{|y| \leq \frac{2}{t}} \frac{C(g)t}{x} |f_+(y)| dy + \int_{\frac{1}{t} \leq |y| \leq \frac{2}{t}} \frac{C(g)t}{x|y|} |f_+(y)| dy \leq C(g, u_1) \frac{\sqrt{t}}{x}.$$

For $0 < \alpha < \beta < 1$ we split integral I_3 into two regions, $|y| \leq t^{-\beta}$ and $t^{-\beta} \leq |y| \leq \frac{2}{t}$. On the first region we upper-bound the integral in s simply by $C\|g\|_{L^\infty}$ and on the other region we use (30) with $\alpha > 0$

$$|I_3| \leq C(g) \int_{|y| \leq t^{-\beta}} y^2 t |f_+(y)| dy + C \int_{t^{-\beta} \leq |y| \leq \frac{2}{t}} |f_+(y)| \frac{C(g, \alpha)}{|x|^\alpha |y|} dy.$$

By Cauchy-Schwarz inequality

$$|I_3| \leq C(g, u_1) t^{1-\frac{5}{2}\beta} + C(g, u_1, \alpha) \frac{t^{\frac{\beta}{2}}}{|x|^\alpha} \leq C(g, u_1, \alpha) \left(t^{1-\frac{5}{2}\beta} + t^{\frac{\beta-\alpha}{2}} \left(1 + \frac{\sqrt{t}}{x} \right) \right).$$

We take $\beta = \frac{3}{10}$ and $\alpha = \frac{3}{20}$ so

$$|I_3| \leq C(g, u_1) \left(t^{\frac{1}{4}} + \frac{\sqrt{t}}{x} \right),$$

and the proof of the Lemma is complete. \square

4.2. The existence and properties of $T(0, x)$.

Lemma 4.4. *There exists $C > 0$ such that for all $n \in \mathbb{N}^*$ and $x \neq 0$ there exists $a_1(x), \dots, a_{2n}(x)$ and $R_n(t, x)$ with*

$$|a_j(x)| \leq (C(u_1))^{j-1}, \quad |R_n(t, x)| \leq C(u_1) \left(\frac{\sqrt{t}}{x} + \frac{t}{x^2} + t^{\frac{1}{4}-} \right),$$

and

$$(31) \quad T(t, x) = \sum_{j=1}^{2n} a_j(x) + R_n(t, x) \\ + (-1)^n \Re \int_x^\infty h(t, s_1) \int_{s_1}^\infty \overline{h(t, s_2)} \dots \Re \int_{s_{2n-2}}^\infty h(t, s_{2n-1}) \int_{s_{2n-1}}^\infty \overline{h(t, s_{2n})} T(t, s_{2n}) ds_{2n} \dots ds_1.$$

Proof. We prove the Lemma by recursion on n . The constant C will be the one from Lemma 4.1. Combining (28) with Lemma 4.3 for $g(s) = 1$ we obtain the result for $n = 1$ with

$$a_1(x) = T^\infty, \quad a_2(x) = -\Re N^\infty \int_x^\infty \widehat{f_+} \left(\frac{s}{2} \right) \frac{ds}{s^{ia^2}}.$$

We suppose the result true for n and we shall prove it for $n + 1$. By replacing in (31) the tangent T in the integral by its expression from (28),

$$\begin{aligned}
T(t, x) &= \sum_{j=1}^{2n} a_j(x) + R_n(t, x) \\
&+ (-1)^n \Re \int_x^\infty h(t, s_1) \int_{s_1}^\infty \overline{h(t, s_2)} \dots \times \\
&\times \Re \int_{s_{2n-2}}^\infty h(t, s_{2n-1}) \int_{s_{2n-1}}^\infty \overline{h(t, s_{2n})} \left(T^\infty + \Im N^\infty \int_{s_{2n}}^\infty h(t, s_{2n+1}) ds_{2n+1} \right) ds_{2n} \dots ds_1 \\
&+ (-1)^{n+1} \Re \int_x^\infty h(t, s_1) \int_{s_1}^\infty \overline{h(t, s_2)} \dots \times \\
&\times \Re \int_{s_{2n}}^\infty h(t, s_{2n+1}) \int_{s_{2n+1}}^\infty \overline{h(t, s_{2n+2})} T(t, s_{2n+2}) ds_{2n+2} \dots ds_1.
\end{aligned}$$

Since

$$\tilde{h}(s) = \widehat{f_+} \left(\frac{s}{2} \right) \frac{1}{s i a^2}$$

is an L^1 function, and since Lemma 4.1 yields $h \in L^1$, we can apply Lemma 4.3 in the first iterated integral as many times as needed to replace everywhere h by $i\tilde{h}$. We gather the difference terms with $R_n(t, x)$ and obtain $R_{n+1}(t, x)$. This way we get the result for $n + 1$ with $a_{2n+1}(x)$ given by

$$(-1)^n \Re \int_x^\infty \tilde{h}(s_1) \int_{s_1}^\infty \overline{\tilde{h}(s_2)} \dots \Re \int_{s_{2n-2}}^\infty \tilde{h}(s_{2n-1}) \int_{s_{2n-1}}^\infty \overline{\tilde{h}(s_{2n})} T^\infty ds_{2n} \dots ds_1,$$

and with $a_{2n+2}(x)$ given by

$$\begin{aligned}
&(-1)^{n+1} \Re \int_x^\infty \tilde{h}(s_1) \int_{s_1}^\infty \overline{\tilde{h}(s_2)} \dots \times \\
&\times \Re \int_{s_{2n-2}}^\infty \tilde{h}(s_{2n-1}) \int_{s_{2n-1}}^\infty \overline{\tilde{h}(s_{2n})} \Re N^\infty \int_{s_{2n}}^\infty \tilde{h}(s_{2n+1}) ds_{2n+1} \dots ds_1.
\end{aligned}$$

□

Let us notice that for $C(u_1)$ small enough

$$\sum_{j=1}^{\infty} |a_j(x)| < \infty.$$

We shall prove that there is a limit for $T(t, x)$ as t goes to zero. From Lemma 4.1 it follows that

$$\begin{aligned}
&\left| \Re \int_x^\infty h(t, s_1) \int_{s_1}^\infty \overline{h(t, s_2)} \dots \Re \int_{s_{2n-2}}^\infty h(t, s_{2n-1}) \int_{s_{2n-1}}^\infty \overline{h(t, s_{2n})} T(t, s_{2n}) ds_{2n} \dots ds_1 \right| \\
&\leq (C(u_1))^{2n},
\end{aligned}$$

and from Lemma 4.3 we get

$$\sum_{j=2n}^{\infty} |a_j(x)| \leq \sum_{j=2n}^{\infty} (C(u_1))^{j-1}.$$

For $\|u_1\|_{H^1}$ small enough, $C\|u_1\|_{H^1} < 1$, so we can choose $n \in \mathbb{N}$ large enough such that

$$\sum_{j=2n}^{\infty} (C\|u_1\|_{H^1})^j \leq C(u_1) \left(\frac{\sqrt{t}}{x} + \frac{t}{x^2} + t^{\frac{1}{4}-} \right).$$

By Lemma 4.4 we conclude

$$\left| T(t, x) - \sum_{j=1}^{\infty} a_j(x) \right| \leq C(u_1) \left(\frac{\sqrt{t}}{x} + \frac{t}{x^2} + t^{\frac{1}{4}-} \right),$$

and in particular $T(t, x)$ has a limit at $t = 0$,

$$T(0, x) = \sum_{j=1}^{\infty} a_j(x),$$

with

$$(32) \quad |T(t, x) - T(0, x)| \leq C(u_1) \left(\frac{\sqrt{t}}{x} + \frac{t}{x^2} + t^{\frac{1}{4}-} \right).$$

We notice that in view of the expression of $a_j(x)$ and of the fact that $\|\hat{f}\|_{L^1} < C\|u_1\|_{H^1} < 1$, we obtain $T(0) \in L^\infty$ and $T_s(0) \in L^2$. Finally, from (22) and (32) we conclude that $T(0, x)$ has a limit as x goes to infinity, and

$$\lim_{x \rightarrow \infty} T(0, x) = T^\infty.$$

Now we focus on $\tilde{N}(t, x)$ as t goes to zero. Estimates (23) and (27) allows us to write for \tilde{N} an estimate similar to (28),

$$(33) \quad \left| \tilde{N}(t, x) - N^\infty + iT^\infty \int_x^\infty \overline{h(t, s)} ds + i \int_x^\infty \overline{h(t, s)} \Im \int_s^\infty h(t, s') \tilde{N}(t, s') ds' ds \right| \\ \leq C(u_1) \left(\frac{\sqrt{t}}{x} + \frac{t}{x^2} + \sqrt{t} \right).$$

Arguing as above for T we obtain a limit for $\tilde{N}(t, x)$ for $x > 0$ as t goes to zero, with

$$(34) \quad |\tilde{N}(t, x) - \tilde{N}(0, x)| \leq C(u_1) \left(\frac{\sqrt{t}}{x} + \frac{t}{x^2} + t^{\frac{1}{4}-} \right).$$

Also, (25) combined with (34) implies that $\tilde{N}(0, x)$ has a limit as x goes to infinity, and

$$\lim_{x \rightarrow \infty} \tilde{N}(0, x) = N^\infty.$$

5. THE SELF-SIMILAR STRUCTURE

We denote

$$T_n(s) = T(t_n, \sqrt{t_n} s) \quad , \quad N_n(s) = N(t_n, \sqrt{t_n} s),$$

for a sequence t_n of times that tend to zero. We compute

$$T'_n(s) = \sqrt{t_n} \Re(\bar{\psi}(t_n, \sqrt{t_n} s) N_n(s)) = \Re\left(ae^{i\frac{s^2}{4}} N_n(s)\right) + o(t_n)N_n(s),$$

$$N'_n(s) = -\sqrt{t_n} \psi(t_n, \sqrt{t_n} s) T_n(s) = -ae^{i\frac{s^2}{4}} T_n(s) + o(t_n)T_n(s).$$

Let us recall that T and N are bounded by 1 and by 2 respectively. It follows that $\mathcal{A} = \{T_n, n \in \mathbb{N}\}$ is a collection of pointwise bounded and equicontinuous functions. Then Arzela-Ascoli theorem allows us to obtain a subsequence, that for simplicity we shall denote again T_n , that converges uniformly on any compact subset of \mathbb{R} . We can do the same for $\mathcal{B} = \{N_n, n \in \mathbb{N}\}$ and conclude that

$$\lim_{n \rightarrow \infty} (T_n(s), N_n(s)) = (T_*(s), N_*(s)).$$

The system satisfied by $(T_*(s), N_*(s))$ is then

$$\begin{cases} T'_*(s) = \Re\left(ae^{i\frac{s^2}{4}} N_*(s)\right), \\ N'_*(s) = ae^{i\frac{s^2}{4}} T_*(s), \end{cases}$$

with initial data $(T_*(0), N_*(0))$, which means that

$$\left(T_*(s), \Re\left(e^{-i\frac{s^2}{4}} N_*(s)\right), \Im\left(e^{-i\frac{s^2}{4}} N_*(s)\right)\right)$$

is the Frenet frame of the curve with curvature and torsion $(a, \frac{s}{2})$, exactly the one of the self-similar profile, see [18]. Hence on the one hand, modulo a rotation,

$$T_*(s) = A_a^+ + \mathcal{O}\left(\frac{1}{s}\right) \quad . \quad N_*(s) = B_a^+ + \mathcal{O}\left(\frac{1}{s}\right).$$

On the other hand, using (32)

$$T_*(s) = \lim_{n \rightarrow \infty} T_n(s) = \lim_{n \rightarrow \infty} (T(t_n, \sqrt{t_n} s) - T(0, \sqrt{t_n} s) + T(0, \sqrt{t_n} s)) = \mathcal{O}\left(\frac{1}{s}\right) + \lim_{n \rightarrow \infty} T(0, \sqrt{t_n} s),$$

so we obtain the existence and the value of $T(0, 0^+)$,

$$T(0, 0^+) = A_a^+.$$

In the same manner we get modulo the same rotation that $T(0, 0^-) = A_a^-$, so we recover at time zero the angle of the self-similar solution.

Similarly we obtain the existence and the values of $\tilde{N}(0, 0^+)$, $\tilde{N}(0, 0^-)$

$$\tilde{N}(0, 0^+) = B_a^+, \quad \tilde{N}(0, 0^-) = B_a^-,$$

where B_a^\pm comes from the self-similar situation and were explicitly described in [18] in terms of the parameter a (see formula (55), (47), (48) and (69)).

6. APPENDIX: THE J-EVOLUTION

At the linear level, if $w(t) = S(t, t_0)w(t_0)$ is the solution of⁴

$$(35) \quad iw_t + w_{xx} + \frac{a^2}{2t}(w + \overline{w}) = 0,$$

with initial data $w(t_0)$ at time t_0 , then $v(t) = J(t)w(t) = (x + 2it\partial_x)w(t)$ satisfies

$$iv_t + v_{xx} + \frac{a^2}{2t}(v + \overline{v}) = \frac{a^2}{2t}(\overline{Jw} - J\overline{w}) = -2ia^2 \overline{w}_x,$$

with initial data $v(t_0) = J(t_0)w(t_0)$ at time t_0 .

We recall that for the free Schrödinger equation, the norm $\|J(t)e^{it\partial_x^2}f\|_{L^2}$ is constant in time, since $J(t)$ commutes with $e^{it\partial_x^2}$. In here, we do not hope such property for (35), but nevertheless we shall get a control in time better than t .

First we shall prove a growth control in time of the Fourier modes of solutions of (35), that improve the one in Lemma 2.1 of [5]. More precisely, the parameter a will not be intervenient anymore in the polynomial control in time of the growth of the Fourier modes.

6.1. Improvement of the growth of the Fourier modes for the linear equation.

Lemma 6.1. *Let $1 \leq t_0 \leq t$. For all $\delta > 0$ there exists a constant $C(a, \delta)$ such that*

$$|\hat{w}(t, \xi)| \leq C(a, \delta) \frac{t^\delta}{t_0^\delta} (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|) \quad \forall \xi \in \mathbb{R}.$$

Proof. We have

$$(36) \quad \partial_t \widehat{\Re w}(t, \xi) = \xi^2 \widehat{\Im w}(t, \xi),$$

$$(37) \quad \partial_t \widehat{\Im w}(t, \xi) = -\xi^2 \widehat{\Re w}(t, \xi) + \frac{a^2}{t} \widehat{\Re w}(t, \xi),$$

so

$$(38) \quad \partial_t^2 \widehat{\Re w}(t, \xi) = \xi^2 \left(-\xi^2 + \frac{a^2}{t} \right) \widehat{\Re w}(t, \xi).$$

We infer

$$\widehat{\Re w}(t, \xi) = \widehat{\Re w}(t_0, \xi) + (t - t_0) \xi^2 \widehat{\Im w}(t_0, \xi) + \int_{t_0}^t (t - \tau) \xi^2 \left(-\xi^2 + \frac{a^2}{\tau} \right) \widehat{\Re w}(\tau, \xi) d\tau.$$

⁴In [5] we have actually denoted by $u(t) = S(t, t_0)w(t_0)$ the solution of

$$iu_t + u_{xx} + \frac{a^2}{t^{1+2ia^2}} \overline{u} = 0,$$

with initial data $w(t_0)$ at time t_0 , so $u(t) = e^{-ia^2 \log t} w(t)$. Therefore getting estimates on $|\hat{u}(t)|$ and $|J(t)\widehat{u}(t)|$ is equivalent to getting estimates on $|\hat{w}(t)|$ and $|J(t)\widehat{w}(t)|$ respectively.

Let $\delta > 0$, and let $0 < \epsilon < \min\{1, a^2\}$ to be chosen also small enough with respect to δ . Then for $\xi^2 \leq \frac{\epsilon}{t}$, if this situation is possible,

$$\begin{aligned} t^{-\delta} |\widehat{\Re w}(t, \xi)| &\leq t^{-\delta} \left(|\widehat{\Re w}(t_0, \xi)| + |\widehat{\Im w}(t_0, \xi)| \right) + t^{-\delta} C a^2 \epsilon \int_{t_0}^t \frac{\tau^\delta}{\tau} d\tau \sup_{t_0 \leq \tau \leq t} \tau^{-\delta} |\widehat{\Re w}(\tau, \xi)| \\ &\leq t_0^{-\delta} \left(|\widehat{\Re w}(t_0, \xi)| + |\widehat{\Im w}(t_0, \xi)| \right) + \frac{C a^2 \epsilon}{\delta} \sup_{t_0 \leq \tau \leq t} \tau^{-\delta} |\widehat{\Re w}(\tau, \xi)| \end{aligned}$$

Then, by choosing ϵ small with respect to δ , we obtain

$$|\widehat{\Re w}(t, \xi)| \leq C(a, \delta) \frac{t^\delta}{t_0^\delta} (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|).$$

Using similar arguments for the imaginary part we get for $\xi^2 \leq \frac{\epsilon}{t}$,

$$\begin{aligned} t^{-\delta} |\widehat{\Im w}(t, \xi)| &\leq t^{-\delta} |\widehat{\Im w}(t_0, \xi)| + t^{-\delta} C \int_{t_0}^t \frac{a^2}{\tau} C(a, \delta) \frac{\tau^\delta}{t_0^\delta} (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|) d\tau \\ &\leq t_0^{-\delta} |\widehat{\Im w}(t_0, \xi)| + t_0^{-\delta} C(a, \delta) (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|), \end{aligned}$$

so

$$|\hat{w}(t, \xi)| \leq C(a, \delta) \frac{t^\delta}{t_0^\delta} (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|),$$

and the Lemma follows for $\xi^2 \leq \frac{\epsilon}{t}$. This part improves Lemma 2.1 in [5], where the control was of $\frac{t^\alpha}{t_0^\alpha}$.

The proof of Lemma 2.2 in [5] contains the result that in the remaining regions $\frac{\epsilon}{t} \leq \xi^2 \leq \frac{2a^2}{t}$ and $\frac{2a^2}{t} \leq \xi^2$ the evolution of the ξ -Fourier modes stays bounded. For instance, when $\frac{\epsilon}{t} \leq \xi^2 \leq \frac{2a^2}{t}$ we did an energy estimate by considering

$$\partial_t \left(|\widehat{\Re w}(t, \xi)|^2 + |\widehat{\Im w}(t, \xi)|^2 \right) = \frac{4a^2}{t} \Re(\widehat{\Re w}(t, \xi) \overline{\widehat{\Im w}(t, \xi)}) \leq \frac{2a^2}{t} \left(|\widehat{\Re w}(t, \xi)|^2 + |\widehat{\Im w}(t, \xi)|^2 \right).$$

By integrating from any $\frac{\epsilon}{\xi^2} \leq t_1 \leq \frac{2a^2}{\xi^2}$ to any $\frac{\epsilon}{\xi^2} \leq t \leq \frac{2a^2}{\xi^2}$, we obtain

$$|\hat{w}(t, \xi)|^2 + |\hat{w}(t, -\xi)|^2 \leq C(a) (|\hat{w}(t_1, \xi)|^2 + |\hat{w}(t_1, -\xi)|^2),$$

so the Lemma follows for $\xi^2 \leq \frac{2a^2}{t}$. For larger times $t \geq \frac{2a^2}{\xi^2}$, we obtained in [5] that the evolution of the Fourier modes is bounded by diagonalizing the system of equations of $\widehat{\Re w}$ and $\widehat{\Im w}$. Therefore the Lemma follows for all ξ . \square

Finally, recall that Lemma 2.2 in [5] asserts that

$$|\hat{w}(t, \xi)| \leq \left(C(a) + C(a, \delta) \frac{1}{(\xi^2 t_0)^\delta} \right) (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|), \quad \forall \xi \neq 0.$$

6.2. J-evolution for the linear equation. Now we turn to the $J(t)u(t)$ evolution. By using the Duhamel formula for $S(t, t_0)$ given by equation (35),

$$(39) \quad v(t) = S(t, t_0)v(t_0) + \int_{t_0}^t S(t, \tau)(-2ia^2 \overline{w}_x(\tau)) d\tau,$$

a similar estimate is obtained also on v ,

$$(40) \quad \begin{aligned} |\hat{v}(t, \xi)| &\leq C(a, \delta) \frac{t^\delta}{t_0^\delta} (|\hat{v}(t_0, \xi)| + |\hat{v}(t_0, -\xi)|) + \int_{t_0}^t C(a, \delta) \frac{t^\delta}{\tau^\delta} |\xi| \frac{\tau^\delta}{t_0^\delta} (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|) d\tau \\ &\leq C(a, \delta) \frac{t^\delta}{t_0^\delta} (|\hat{v}(t_0, \xi)| + |\hat{v}(t_0, -\xi)|) + C(a, \delta) \frac{t^\delta}{t_0^\delta} t |\xi| (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|), \end{aligned}$$

and we finally obtain

$$(41) \quad \|\hat{v}(t, \xi)\|_{L^2(\xi^2 \leq \frac{1}{t})} \leq C(a, \delta) \frac{t^\delta}{t_0^\delta} \|v(t_0)\|_{L^2} + C(a, \delta) \frac{t^{\delta+\frac{1}{2}}}{t_0^\delta} \|w(t_0)\|_{L^2}.$$

On the other hand, we get the following version of Lemma 2.2 in [5].

Lemma 6.2. *For all $\xi \neq 0$ and $1 \leq t_0 \leq t$ the following estimate holds*

$$(42) \quad \begin{aligned} |\hat{v}(t, \xi)| &\leq \left(C(a) + \frac{C(a, \delta)}{(\xi^2 t_0)^\delta} \right) (|\hat{v}(t_0, \xi)| + |\hat{v}(t_0, -\xi)|) \\ &+ \left(C(a) + C(a, \delta) \frac{1 + |\log |\xi||}{(\xi^2 t_0)^\delta} \right) \frac{|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|}{|\xi|}. \end{aligned}$$

Proof. For $\xi^2 \lesssim \frac{1}{t}$ the Lemma follows from (40). For $\xi^2 \geq \frac{2a^2}{t}$ we shall diagonalize the system

$$(43) \quad \partial_t \widehat{\mathfrak{R}v}(t, \xi) = \xi^2 \widehat{\mathfrak{S}v}(t, \xi) - 2ia^2 \xi \widehat{\mathfrak{R}w}(t, \xi),$$

$$(44) \quad \partial_t \widehat{\mathfrak{S}v}(t, \xi) = -\xi^2 \widehat{\mathfrak{R}v}(t, \xi) + \frac{a^2}{t} \widehat{\mathfrak{R}v}(t, \xi) + 2ia^2 \xi \widehat{\mathfrak{S}w}(t, \xi).$$

With similar notations as in [5], we denote for $t \geq 2a^2$

$$\begin{aligned} A(t, \xi) &= \widehat{\mathfrak{R}v}\left(\frac{t}{\xi^2}, \xi\right), \quad B(t, \xi) = \widehat{\mathfrak{S}v}\left(\frac{t}{\xi^2}, \xi\right), \\ Y(t, \xi) &= \widehat{\mathfrak{R}w}\left(\frac{t}{\xi^2}, \xi\right), \quad Z(t, \xi) = \widehat{\mathfrak{S}w}\left(\frac{t}{\xi^2}, \xi\right), \end{aligned}$$

so we have the system

$$(45) \quad \begin{cases} \partial_t A(t, \xi) = B(t, \xi) - \frac{2ia^2}{\xi} Y(t, \xi), \\ \partial_t B(t, \xi) = \left(-1 + \frac{a^2}{t}\right) A(t, \xi) + \frac{2ia^2}{\xi} Z(t, \xi). \end{cases}$$

We shall diagonalize the system

$$\partial_t \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\left(1 - \frac{a^2}{t}\right) & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + \frac{2ia^2}{\xi} \begin{pmatrix} -Y \\ Z \end{pmatrix}.$$

Let

$$\alpha(t) = \sqrt{1 - \frac{a^2}{t}} \quad , \quad P(t) = \begin{pmatrix} 1 & 1 \\ i\alpha(t) & -i\alpha(t) \end{pmatrix}.$$

In particular,

$$\frac{1}{\sqrt{2}} \leq \alpha(t) \leq 1 \quad , \quad P^{-1}(t) = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2\alpha(t)} \\ \frac{1}{2} & \frac{i}{2\alpha(t)} \end{pmatrix}.$$

Then the new functions

$$\begin{pmatrix} A_1(t, \xi) \\ B_1(t, \xi) \end{pmatrix} = P^{-1}(t) \begin{pmatrix} A(t, \xi) \\ B(t, \xi) \end{pmatrix}$$

satisfy

$$\partial_t \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \partial_t(P^{-1})P \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} + \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} + P^{-1} \frac{2ia^2}{\xi} \begin{pmatrix} -Y \\ Z \end{pmatrix}.$$

We introduce

$$\Phi(t) = t - \frac{a^2}{2} \log t - \int_t^\infty \alpha(s) - 1 + \frac{a^2}{2s} ds,$$

that verifies

$$\Phi(t)' = \alpha(t).$$

Finally, the functions

$$\begin{pmatrix} A_2(t, \xi) \\ B_2(t, \xi) \end{pmatrix} = \begin{pmatrix} e^{-i\Phi(t)} & 0 \\ 0 & e^{i\Phi(t)} \end{pmatrix} \begin{pmatrix} A_1(t, \xi) \\ B_1(t, \xi) \end{pmatrix}$$

are solutions of

$$\begin{aligned} \partial_t \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} &= M(t) \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} + \begin{pmatrix} e^{-i\Phi(t)} & 0 \\ 0 & e^{i\Phi(t)} \end{pmatrix} P^{-1} \frac{2ia^2}{\xi} \begin{pmatrix} -Y \\ Z \end{pmatrix} \\ &= M(t) \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} + \frac{2ia^2}{\xi} \begin{pmatrix} e^{-i\Phi(t)}(Y - \frac{i}{2\alpha}Z) \\ e^{i\Phi(t)}(Y + \frac{i}{2\alpha}Z) \end{pmatrix} \end{aligned}$$

where

$$M(t) = \begin{pmatrix} e^{-i\Phi(t)} & 0 \\ 0 & e^{i\Phi(t)} \end{pmatrix} \partial_t(P^{-1})P \begin{pmatrix} e^{i\Phi(t)} & 0 \\ 0 & e^{-i\Phi(t)} \end{pmatrix} = \frac{a^2}{4t^2\alpha^2} \begin{pmatrix} -1 & e^{-2i\Phi(t)} \\ e^{2i\Phi(t)} & -1 \end{pmatrix}.$$

By the relation (31) in [5], for $t \geq 12a^2$,

$$(46) \quad \partial_t \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}(t, \xi) = M(t) \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}(t, \xi) + \frac{2ia^2}{\xi} \left(\begin{pmatrix} -e^{-2i\Phi(t)} Z^+(\xi) \\ -e^{2i\Phi(t)} Y^+(\xi) \end{pmatrix} + R(t, \xi) \right),$$

where

$$\overline{Y^+(-\xi)} = Z^+(\xi) = \frac{1}{2} e^{-i\frac{a^2}{2} \log \xi^2} \hat{u}_+(\xi),$$

and

$$R(t, \xi) = \begin{pmatrix} -e^{-i\Phi(t)} \int_t^\infty \frac{ia^2 e^{i\Phi(\tau)}}{2\alpha^3(\tau) \tau^2} Z(\tau, \xi) d\tau \\ e^{i\Phi(t)} \int_t^\infty \frac{ia^2 e^{-i\Phi(\tau)}}{2\alpha^3(\tau) \tau^2} Z(\tau, \xi) d\tau \end{pmatrix}.$$

For $2a^2 \leq \tilde{t} \leq t$ we integrate by parts again. We do it just for the first component of $R(t, \xi)$ because the other one is similar. We obtain

$$\begin{aligned} \int_{\tilde{t}}^t -e^{-i\Phi(\tau)} \int_\tau^\infty \frac{ia^2 e^{i\Phi(\theta)}}{2\alpha^3(\theta) \theta^2} Z(\theta, \xi) d\theta d\tau &= \left[e^{-i\Phi(\tau)} \frac{1}{i\alpha(\tau)} \int_\tau^\infty \frac{ia^2 e^{i\Phi(\theta)}}{2\alpha^3(\theta) \theta^2} Z(\theta, \xi) d\theta \right]_{\tilde{t}}^t \\ &- \int_{\tilde{t}}^t e^{-i\Phi(\tau)} \frac{a^2}{i2\alpha^3(\tau) \tau^2} \int_\tau^\infty \frac{ia^2 e^{i\Phi(\theta)}}{2\alpha^3(\theta) \theta^2} Z(\theta, \xi) d\theta d\tau + \int_{\tilde{t}}^t e^{-i\Phi(\tau)} \frac{1}{i\alpha(\tau)} \frac{ia^2 e^{i\Phi(\tau)}}{2\alpha^3(\tau) \tau^2} Z(\tau, \xi) d\tau. \end{aligned}$$

From Lemma 2.2 in [5] it follows that we are in the region where $Z(\tau, \xi)$ is bounded by $C(a) (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|)$. Moreover, $\frac{1}{\sqrt{2}} \leq \alpha(t) \leq 1$, so

$$\left| \int_{\tilde{t}}^t -e^{-i\Phi(\tau)} \int_\tau^\infty \frac{ia^2 e^{i\Phi(\theta)}}{2\alpha^3(\theta) \theta^2} Z(\theta, \xi) d\theta d\tau \right| \leq \frac{C(a)}{t} (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|).$$

Again since $\frac{1}{\sqrt{2}} \leq \alpha(t) \leq 1$, all the entries of $M(t)$ are upper-bounded by $\frac{a^2}{2t^2}$. In conclusion, integrating expression (46), we have for $2a^2 \leq \tilde{t} \leq t$

$$\begin{aligned} |A_2(t, \xi)| + |B_2(t, \xi)| &\leq |A_2(\tilde{t}, \xi)| + |B_2(\tilde{t}, \xi)| + \int_{\tilde{t}}^t \frac{a^2}{t^2} (|A_2(\tau, \xi)| + |B_2(\tau, \xi)|) d\tau \\ &+ \frac{C(a)}{|\xi|} (|\hat{u}_+(\xi)| + |\hat{u}_+(-\xi)|) + \frac{C(a)}{t|\xi|} (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|). \end{aligned}$$

So we get

$$|A_2(t, \xi)| + |B_2(t, \xi)| \leq 2 (|A_2(\tilde{t}, \xi)| + |B_2(\tilde{t}, \xi)|) + \frac{C(a)}{|\xi|} |\hat{u}_+(\xi)| + \frac{C(a)}{t|\xi|} (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|).$$

Finally, from the relation

$$|A_2|^2 + |B_2|^2 = \left| \frac{1}{2}A - \frac{i}{2\alpha}B \right|^2 + \left| \frac{1}{2}A + \frac{i}{2\alpha}B \right|^2 = \frac{1}{2}|A|^2 + \frac{1}{2\alpha^2}|B|^2,$$

and from $\frac{1}{\sqrt{2}} \leq \alpha(t) \leq 1$ it follows that for $2a^2 \leq \tilde{t} \leq t$,

$$|A(t, \xi)|^2 + |B(t, \xi)|^2 \leq C(|A(\tilde{t}, \xi)|^2 + |B(\tilde{t}, \xi)|^2) + \frac{C(a)}{|\xi|^2} |\hat{u}_+(\xi)|^2 + \frac{C(a)}{t^2|\xi|^2} (|\hat{w}(t_0, \xi)| + |\hat{w}(t_0, -\xi)|).$$

By recovering the first variables and using Lemma 2.10 in [5] on the asymptotic state $\hat{u}_+(\xi)$, we obtain the Lemma. \square

The pointwise estimate (42) implies

$$(47) \quad \|\hat{v}(t, \xi)\|_{L^2(\frac{1}{t} \leq \xi^2)} \leq C(a, \delta) \frac{t^\delta}{t_0^\delta} \|v(t_0)\|_{L^2} + C(a, \delta) \frac{t^{\delta+\frac{1}{2}+}}{t_0^\delta} \|w(t_0)\|_{L^2}.$$

In conclusion, gathering (41) and (47), we obtain a control for the L^2 norm of the J -evolution of the linear solutions,

$$(48) \quad \|J(t)S(t, t_0)f\|_{L^2} \leq C(a, \delta) \frac{t^\delta}{t_0^\delta} \|J(t_0)f\|_{L^2} + C(a, \delta) \frac{t^{\delta+\frac{1}{2}+}}{t_0^\delta} \|f\|_{L^2}.$$

6.3. J-evolution for the nonlinear equation. We want to show by a bootstrap argument that the solution of the nonlinear equation

$$iu_t + u_{xx} + \frac{a+u}{2t}(|a+u|^2 - a^2) = 0$$

enjoys a good control in time of $\|J(t)u(t)\|_{L^2}$. First, let us mention that this quantity is finite in time. Indeed, $u(t) \in \dot{H}^1$ and it was proved in Lemma B.1 in [5] that $xu(t) \in L^2$ with a high polynomial growth in time.

Proposition 6.3. *If $xu_1 \in L^2$ and if u_1 is small enough in X_1^γ , then for all $t \geq 1$ we have*

$$\|J(t)u(t)\|_{L^2} \leq C(u_1) t^{\frac{3}{4}},$$

Proof. The solution of the nonlinear equation writes as

$$(49) \quad u(t, x) = S(t, 1)u_1 + \int_1^t S(t, \tau) \frac{iF(\tau)}{\tau} d\tau.$$

with $F(u)$ given by

$$(50) \quad F(u) = \frac{|u|^2 u + a(u^2 + 2|u|^2)}{2t}.$$

We have from (48)

$$t^{-\frac{3}{4}} \|J(t)S(t, 1)u_1\|_{L^2} \leq C(u_1)$$

provided that we choose $\delta < \frac{1}{4}$. Then the worst Duhamel term is the quadratic one. We use again (48) with $\delta < \frac{1}{4}$,

$$\begin{aligned} & t^{-\frac{3}{4}} \left\| J(t) \int_1^t S(t, \tau) u^2(\tau) \frac{d\tau}{\tau} \right\|_{L^2} \\ & \leq C(a, \delta) t^{-\frac{3}{4}} \int_1^t \left(\frac{t^\delta}{\tau^\delta} \|J(\tau)u^2(\tau)\|_{L^2} + \log t \frac{t^{\delta+\frac{1}{2}+}}{\tau^\delta} \|u^2(\tau)\|_{L^2} \right) \frac{d\tau}{\tau}. \end{aligned}$$

Here $J(t)$ acts on a non-gauge invariant power, so we have to split this term into weight and derivative part, and loose a t -power. By using Cauchy-Schwarz inequality

$$t^{-\frac{3}{4}} \left\| J(t) \int_1^t S(t, \tau) u^2(\tau) \frac{d\tau}{\tau} \right\|_{L^2} \leq C(a, \delta) t^{-\frac{3}{4}} t^\delta \int_1^t \|xu^2(\tau)\|_{L^2} \frac{d\tau}{\tau^{1+\delta}}$$

$$\begin{aligned}
& + C(a, \delta) t^{-\frac{3}{4}} t^\delta \int_1^t \|u_x(\tau)u(\tau)\|_{L^2} \frac{d\tau}{\tau^\delta} + C(a, \delta) t^{-\frac{3}{4}} \log t t^{\delta+\frac{1}{2}^+} \|u\|_{L^\infty L^2} \|u\|_{L^\infty L^\infty} \\
& \leq C(a, \delta) \sup_{1 \leq \tau \leq t} \|\tau^{-\frac{3}{4}} J(\tau)u(\tau)\|_{L^2} \|u\|_{L^\infty H^1} + C(a, \delta) \|u_x\|_{L^8 L^4} \|u\|_{L^8 L^4} \\
& \quad + C(a, \delta) t^{-\frac{3}{4}} \log t t^{\delta+\frac{1}{2}^+} \|u\|_{L^\infty H^1}^2.
\end{aligned}$$

In [5] it was shown that for small initial data $u_1 \in X_1^\gamma$, the solution u satisfies

$$u \in L^\infty(1, \infty) L^2 \cap L^4(1, \infty) L^\infty,$$

and implicitly u belongs to all interpolated Strichartz spaces. So provided that u_1 and $\partial_x u_1$ are small enough in X_1^γ

$$t^{-\frac{3}{4}} \left\| J(t) \int_1^t S(t, \tau) u^2(\tau) \frac{d\tau}{\tau} \right\|_{L^2} \leq \frac{1}{3a} \sup_{1 \leq \tau \leq t} \|\tau^{-\frac{3}{4}} J(\tau)u(\tau)\|_{L^2} + C(u_1).$$

The other quadratic term can be treated the same, and we obtain

$$\begin{aligned}
& t^{-\frac{3}{4}} \left\| J(t) \int_1^t S(t, \tau) \frac{a u^2(\tau)}{2\tau} d\tau \right\|_{L^2} + t^{-\frac{3}{4}} \left\| J(t) \int_1^t S(t, \tau) \frac{a^2 |u|^2(\tau)}{\tau} d\tau \right\|_{L^2} \\
& \leq \frac{1}{3} \sup_{1 \leq \tau \leq t} \|\tau^{-\frac{3}{4}} J(\tau)u(\tau)\|_{L^2} + C(u_1).
\end{aligned}$$

The cubic term is gauge invariant, so by (48) with $\delta < \frac{1}{4}$ we obtain

$$\begin{aligned}
& t^{-\frac{3}{4}} \left\| J(t) \int_1^t S(t, \tau) |u|^2 u(\tau) \frac{d\tau}{\tau} \right\|_{L^2} \\
& \leq C(a, \delta) t^{-\frac{3}{4}} \int_1^t \left(\frac{t^\delta}{\tau^\delta} \|J(\tau)u(\tau)\|_{L^2} \|u(\tau)\|_{L^\infty}^2 + \log t \frac{t^{\delta+\frac{1}{2}^+}}{\tau^\delta} \|u(\tau)\|_{L^2} \|u(\tau)\|_{L^\infty}^2 \right) \frac{d\tau}{\tau}.
\end{aligned}$$

Again providing that u_1 and $\partial_x u_1$ are small enough in X_1^γ ,

$$\begin{aligned}
& t^{-\frac{3}{4}} \left\| J(t) \int_1^t S(t, \tau) |u|^2 u(\tau) \frac{d\tau}{\tau} \right\|_{L^2} \\
& \leq C(a, \delta) \sup_{1 \leq \tau \leq t} \|\tau^{-\frac{3}{4}} J(\tau)u(\tau)\|_{L^2} \|u\|_{L^\infty H^1}^2 + C(a, \delta) t^{-\frac{3}{4}} \log t t^{\delta+\frac{1}{2}^+} \|u\|_{L^\infty H^1}^3 \\
& \leq \frac{1}{6} \sup_{1 \leq \tau \leq t} \|\tau^{-\frac{3}{4}} J(\tau)u(\tau)\|_{L^2} + C(u_1).
\end{aligned}$$

In conclusion, for all $t \geq 1$ we have

$$\sup_{1 \leq \tau \leq t} \|\tau^{-\frac{3}{4}} J(\tau)u(\tau)\|_{L^2} \leq \frac{2}{3} \sup_{1 \leq \tau \leq t} \|\tau^{-\frac{3}{4}} J(\tau)u(\tau)\|_{L^2} + C(u_1),$$

and the Lemma follows. \square

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